NIMEQ: MHD Equilibrium Solver for NIMROD

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Abstract

- A Grad-Shafranov equilibrium solver is developed within the NIMROD framework to create plasma profiles for realistic geometry. The traditional Grad-Shafranov operator is converted to a pure divergence allowing the use of standard regularity conditions for the quantity ψ/R^2 in simply connected domains. The resulting equation is solved in the weak form using a finite element representation. A Picard scheme is used to advance the nonlinear iteration.

Outline

- Introduction
- Outline of solver
  - Derivation of finite element Grad-Shafranov operator
  - Inhomogeneous boundary conditions
  - Nonlinear iteration
  - Calculation of equilibrium fields
  - NIMROD Implementation
- Verification of solver accuracy
  - Constant lambda cylindrical pinch
  - Circular cross section tokamak
  - SSPX geometry
- Conclusions & future work

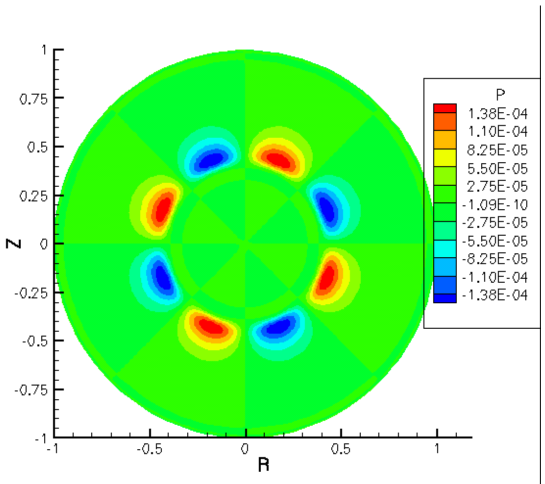
INTRODUCTION: The NIMROD code is a computational laboratory designed to study 3D plasmas for fusion based applications [Phys. Plasmas **10** (2003) 1727].

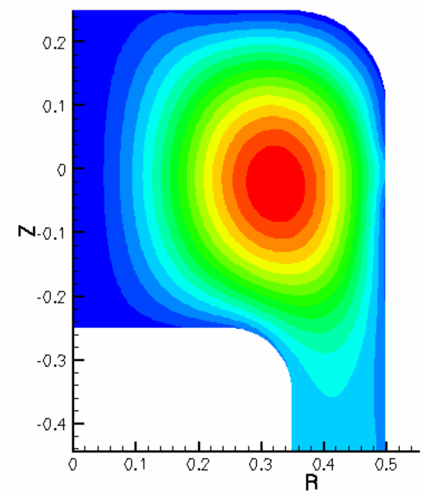
- NIMROD uses a spectral element expansion in 2 dimensions and finite Fourier series in the third.
  - Spectral element expansion provides NIMROD with the flexibility to study plasma in complex geometries.
    - High-order accuracy resolves extreme anisotropy.
  - Fourier expansion in third dimension requires a degree of symmetry.
    - Linear periodic or toroidal geometry
- Nimrod uses an implicit leap frog time advance [JoP: Conf. Ser. **16**, 25 (2005)].
  - Allows timesteps that are large relative to normal-mode propagation times.

NIMROD computations use Grad-Shafranov equilibria in many applications.

- The capability to create simple 1D equilibria exists within NIMROD's preprocessor .
- For more complex geometries, an equilibrium is created from an external source and then interpolated to a NIMROD mesh.
  - The process of interpolation introduces numerical errors into the equilibrium.
  - Software designed to reconstruct equilibria from experimental measurements may sacrifice accuracy for speed.
  - The propagation of these errors though the simulation increases as the disparity of spatial and temporal scales in the physics increase.
- NIMROD simulations use equilibria as initial states or as fixed data, where the code just evolves perturbations.

Simulations used to study linear instabilities in SSPX discharges illustrate the practicality of a native equilibrium solver.





- Cylindrical pinch
  - SSPX profiles have been approximated with linear cylindrical pinches.
  - Analytic solutions can be used to generate equilibrium.
  - Simplifications result in loss of geometric effects.
- Realistic mesh representing entire flux conserver
  - Geometric effects from gun and wall are included.
  - Initial conditions can be read from reconstructed equilibria.
  - A native solve can be used to scan profiles in this realistic geometry.

Creating an equilibrium solver within NIMROD allows the full use of spectral element flexibility and accuracy.

- Using the same expansion to create equilibria and run MHD simulations eliminates all interpolation errors.
  - Best possible equilibrium for a particular grid.
- NIMROD users will be able to modify the solver to meet demands of specific applications.
  - Magnetosphere plasma profiles
  - Parameterized tokamaks, RFPs, and spheromaks
  - Refine equilibria from other codes
- Modularity of NIMROD provides most of the computational tools needed for the equilibrium solver.

Outline of solver: The Grad-Shafranov equation describes 2 dimensional axisymmetric plasma equilibria with no flow.

$$\Delta^*\psi = -F(\psi)F'(\psi) - \mu_0 R^2 p'(\psi)$$

$$\Delta^* \equiv \frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2}$$

- Nonlinear, elliptic, partial differential equation for the flux function $\psi(R,z)$
  - F and p are independent functions of ψ and need to be specified.
  - $F = RB_\varphi$ p=pressure
- In a linear geometry $(\varphi \Rightarrow z)$, the del star operator reduces to the Laplacian in (x,y).

$$\nabla^2\psi = -F(\psi)F'(\psi) - \mu_0 p'(\psi)$$

The Grad-Shafranov operator can be expressed as a total divergence.

$$\vec{A} \equiv R^2 \nabla (R^{-2}\psi) \qquad \Xi = R^{-2}\psi$$

$$\nabla \cdot \vec{A} = \frac{\partial^2}{\partial R^2}\psi - \frac{1}{R}\frac{\partial}{\partial R}\psi + \frac{\partial^2}{\partial z^2}\psi = \Delta^* \psi$$

- Expanding ψ shows that Ξ is well behaved near R=0 as long as ψ_0=0.

$$\psi = \psi_0 + \psi_2 R^2 + \ldots$$

$$\Xi = \frac{\psi_0}{R^2} + \psi_2 + \ldots$$

- We choose ψ_0=0 for the arbitrary constant so that Ξ satisfies standard regularity conditions.

A finite element algebraic equation is formulated for the dependent variable Ξ.

$$\nabla \cdot (R^2 \nabla \Xi) = G(\Xi, R), \quad G(\Xi, R) \equiv -\mu_0 FF' - \mu_0 p' R^2$$

$$\int \left(\nabla \cdot (R^2 \nabla \Xi) \right) \alpha_i (R, z) dV = \int G \alpha_i dV$$

Weak form of Grad-Shafranov equation

$$\Xi = \sum \Xi_j \alpha_j (R, z)$$

Expand dependent variable with finite element basis functions

$$\sum_j \int R^2 \left(\frac{\partial \alpha_i}{\partial R} \frac{\partial \alpha_j}{\partial R} + \frac{\partial \alpha_i}{\partial Z} \frac{\partial \alpha_j}{\partial Z} \right) dV \Xi_j = -\int G \alpha_i dV + surfaceterms$$

No contribution for Dirchlet boundary conditions on Ξ

$$\sum_j M_{ij}^* \Xi_j = g_i$$

A surface integral is performed to calculate ψ along the boundary.

- The change in the flux function along the boundary is given by

$$\Delta\psi = -\int_{x1}^{x2} R(\vec{B}\cdot\hat{n})dl$$

- Normal magnetic field is prescribed.
- Gaussian quadrature is used to perform surface integral.
- The value Ξ along the border is used as essential conditions on the expansion.
- In toroidal geometry regularity places further constraints on ψ.

$$\lim_{r\to 0} B_z/2 \to \psi_2$$

A modified Picard scheme is used to perform nonlinear iteration.

$$\underline{\Xi}^{n+1} = w \underline{\underline{M}}^{-1} \underline{G}^{n} + (1 - w)\underline{\Xi}^{n}$$

- w is a relaxation parameter that provides stabilization to nonlinear iteration.
- Convergence in checked by comparing the residual of the nonlinear solve with the magnitude of

$$\frac{\left\| \Delta^{*}\psi + FF' + \mu_0 R^2 P' \right\|}{\left\| \Delta^{*}\psi \right\|} \equiv error$$

- In practice, Picard scheme is adequate.
  - Computation takes 1-2 minutes on laptops and workstations.
- A Newton iteration can be added to accelerate convergence near solution.

The equilibrium $\vec{B}(R,z)$, $\vec{J}(R,z)$, and $p(R,z)$ are calculated from Ξ, $F(\psi)$, and $p(\psi)$.

$$R\vec{B} = R\nabla\phi \times \nabla\psi + F\hat{e}_\phi$$

$$\mu_0\vec{J} = F'\nabla\psi \times \nabla\phi + \Delta^*\psi\nabla\phi$$

- Chain rule is applied to calculate $\vec{B}$ from the known quantity Ξ and then $\vec{J}_{pol}$ is calculated from $\vec{B}_{pol.}$
  - Nimrod uses RB_φ and $R^{-1}J_\varphi$ in the representation of equilibrium fields.

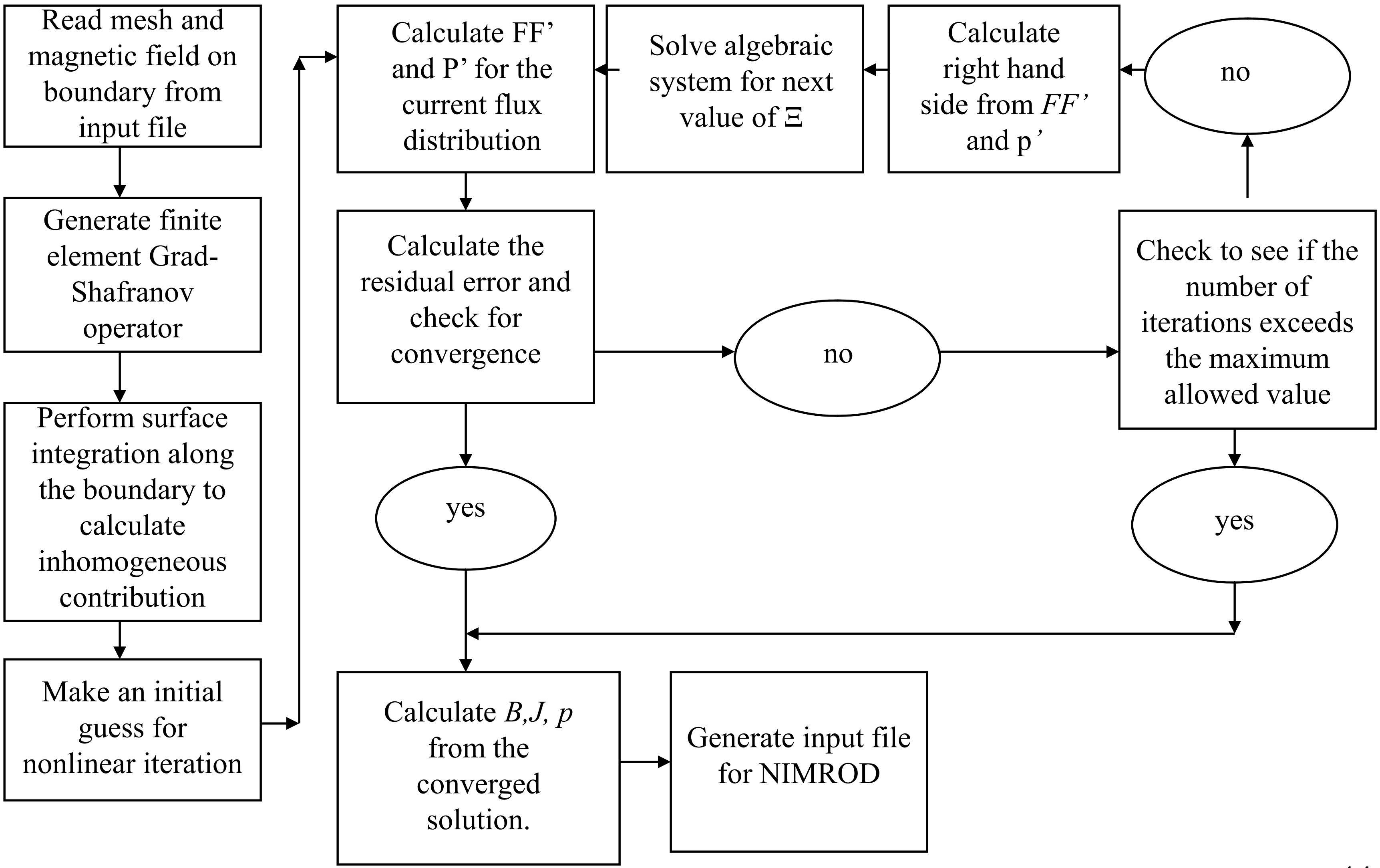
$$R\vec{B}_{pol} = \hat{e}_\phi \times \left(R^2\nabla\Xi + 2R\Xi\hat{e}_r\right)$$

$$RB_\phi = F$$

$$\mu_0\vec{J}_{pol} = -\vec{B}_{pol}F'$$

$$R^{-1}J_\phi = -FF'/\mu_0R^2 - P'$$

NIMEQ flowchart shows the steps in the nonlinear iteration.



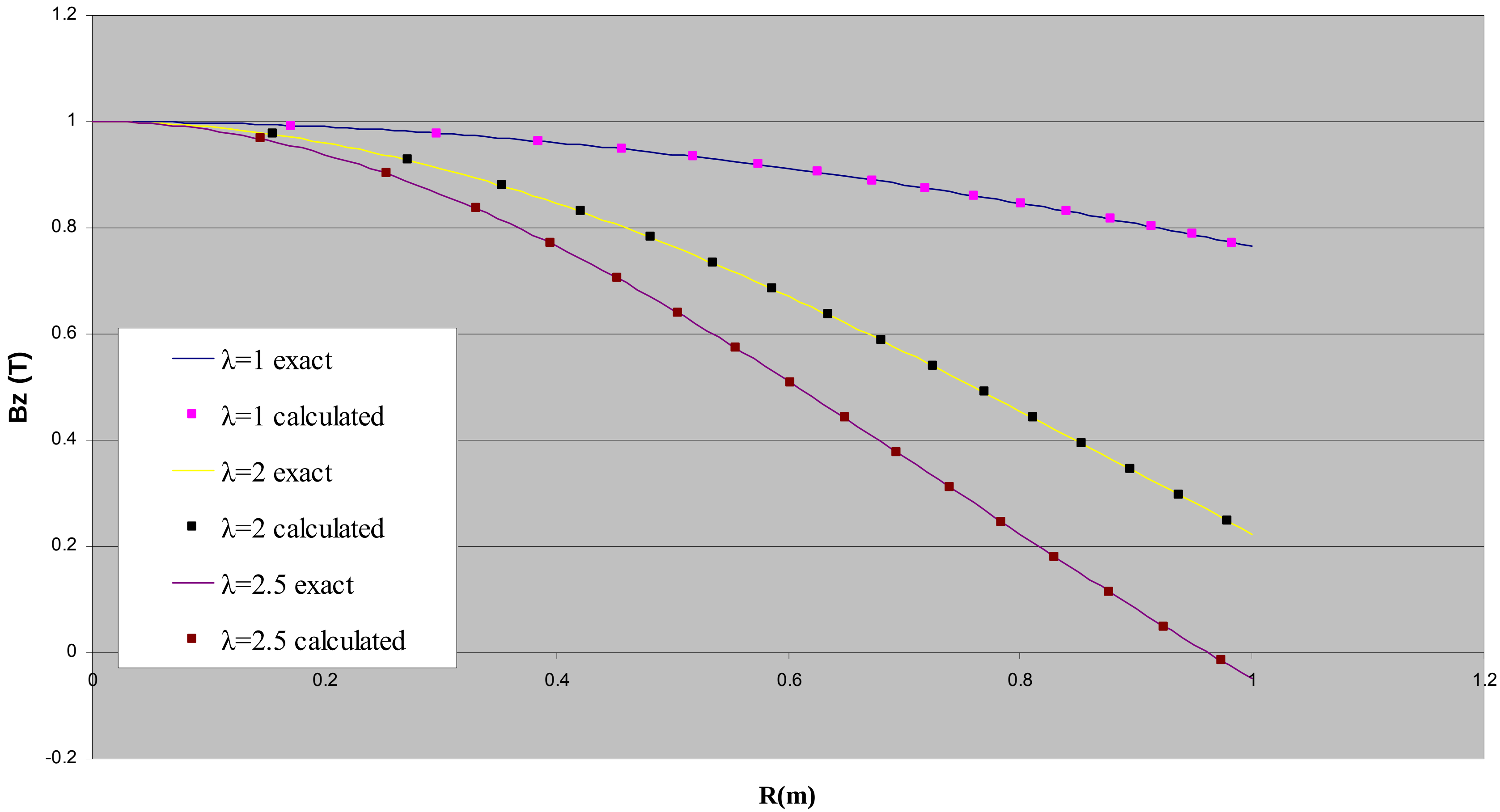
Several basic models for F and p have been implemented.

- A normalized ring flux Y has been calculated to distinguish between physics in open and closed flux regions.
  - Y=0 on the magnetic axis
  - Y=1 on separatrix and open flux region
- Three models for F
  - F=f_open+f1*(1-Y)+4*f2*Y(Y-1)
  - $F = f\_open + (f\_axis - f\_open)*(1-3Y^2+2Y^3)$
  - $F = f0 + f1\psi$
- Two models for pressure
  - $P = P_0$
  - $P = P\_open + (P\_axis - P\_open)*(1-3Y^2+2Y^3)$

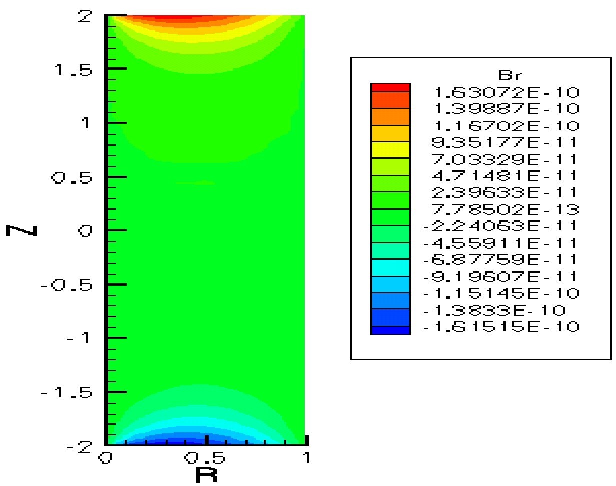
Verification of solver accuracy :Analytic solutions for the magnetic field in a constant lambda cylindrical pinch are used to benchmark solver.

- Magnetic fields are functions radius only.
  - B_φ=-$B_0 J_1(\lambda R)$ B_z=$B_0 J_0(\lambda R)$ B_r=0
  - Nimrod uses (R,z,φ) coordinates instead of (R,θ,z) coordinates.
  - B_z reverses sign at λR=2.404.
- Three different computation grids are used to test different aspects of the solver.
  - Rectangular grid in a toroidal configuration with B specified along the entire boundary.
  - Rectangular grid in a toroidal configuration with periodic boundary conditions in the *z*-direction and the flux specified on the outer boundary.
  - Circular grid with periodicity in the *z-direction* and the flux specified on the outer boundary.

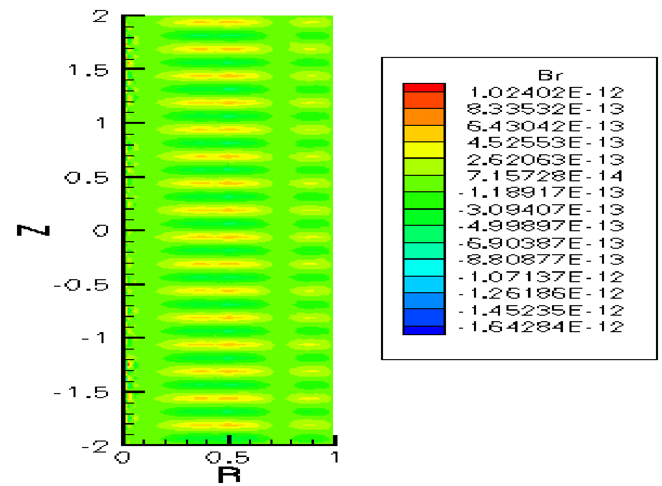
Plots of equilibrium fields generated with a rectangular mesh show agreement with analytic solutions.



Contours of B_R for rectangular grids with different boundary conditions display the surface integrator accuracy.

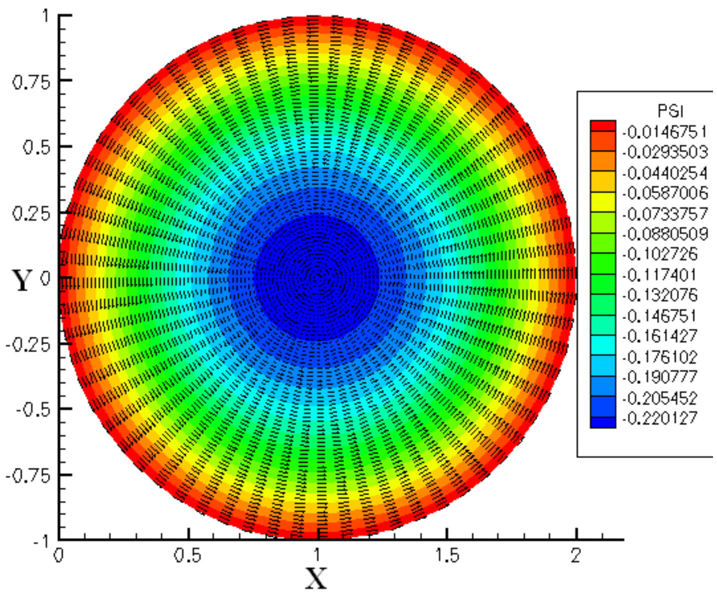


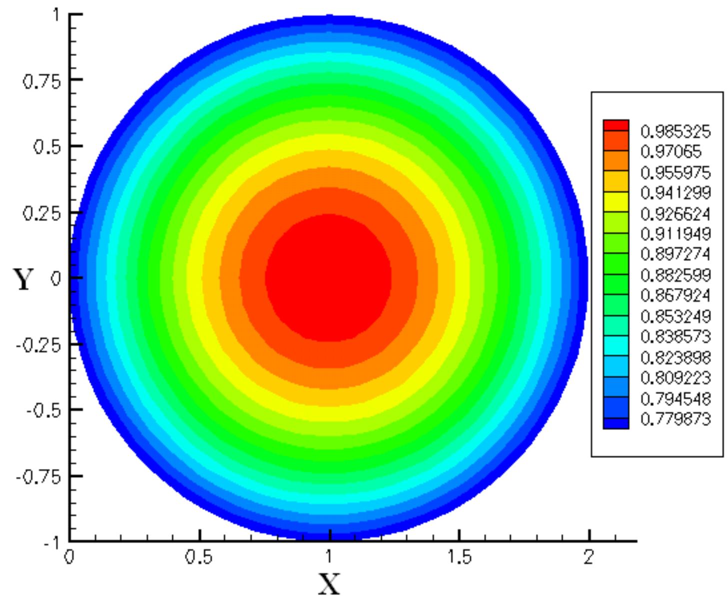
- Values of $\vec{B}$ specified on boundary
- 16x16 elements with 4th order polynomials
- Max B_R =1.863x10^{-10}
- B_0= 1.0



- Periodic boundary conditions in z with ψ specified on outer surface
- 16x16 elements with 4th order polynomials
- Max B_R =1.833x10^{-12}
- B_0= 1.0

Comparison of equilibrium fields on circular grid verifies the solver in non-toroidal systems.

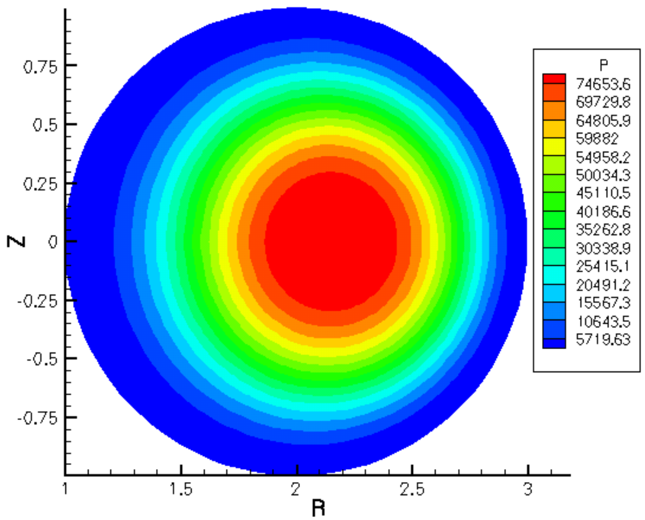




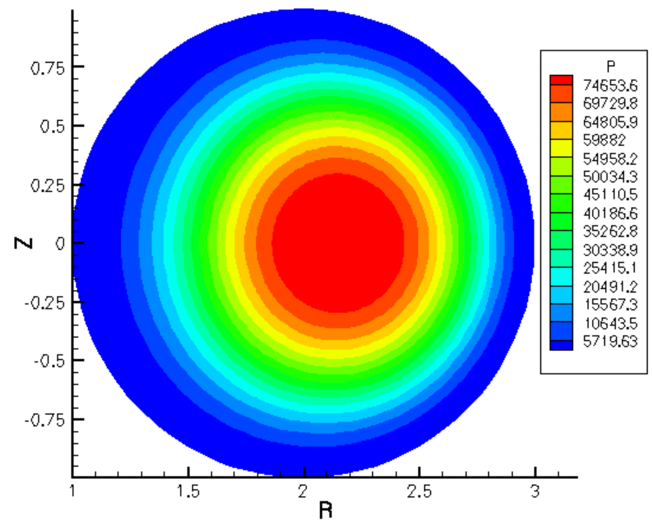
- Fields for λ =1 solutions agree with theoretical values.
- Plots of $\vec{B}_{pol}$ show that that B_R is small and depends only on ψ.
- Nonlinear iterations fail to converge for equilibria with λ =2.5 (axial field reversal).
  - Additional work is being done to improve convergence.

Axisymmetric nonlinear simulations of a circular cross-section tokamak test quality of equilibrium with finite pressure.

Initial pressure



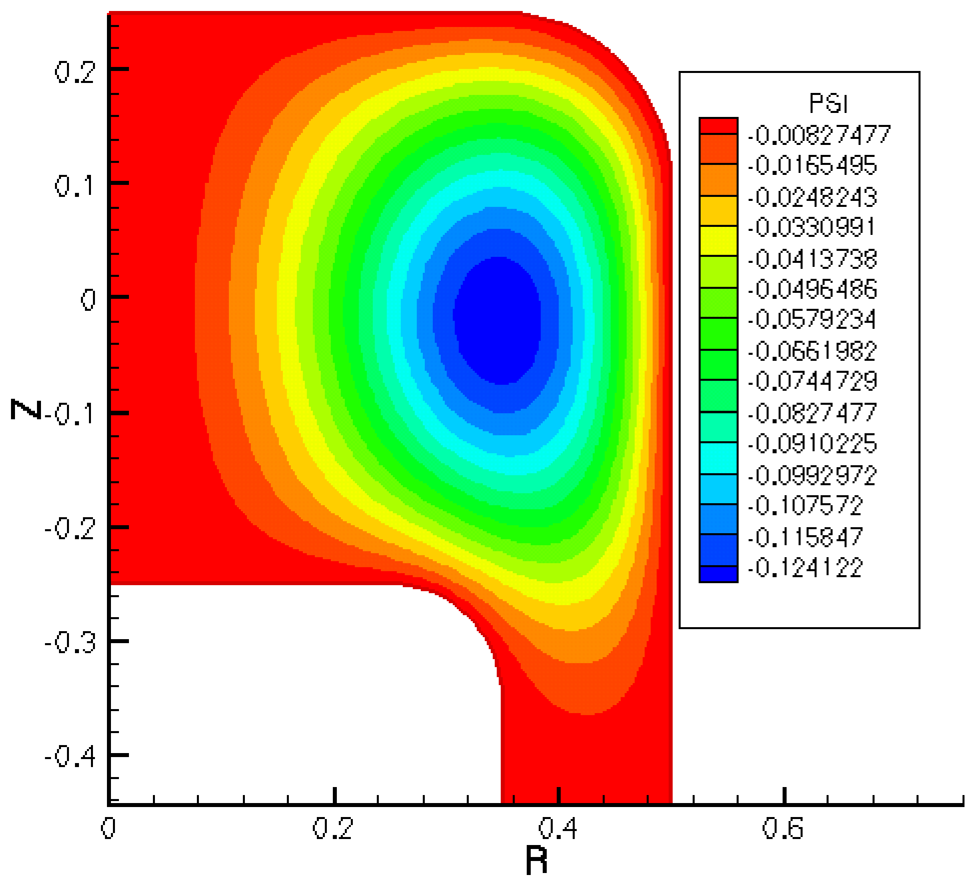
Final pressure



- Generated equilibrium is used as initial condition for nonlinear simulation.
  - Major Radius =2 Minor radius =1
  - Cubic pressure profile with β~0.3
  - 18x18 circular grid with 5th order polynomials
- Simulation ran for 7500 τ_A.
- A change in peak pressure of 0.046% is observed.

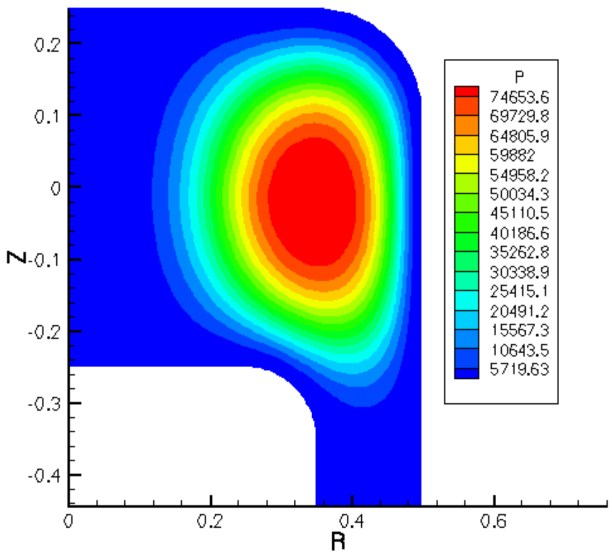
An equilibrium for SSPX is generated on a grid that represents its flux conserver.

- Quadratic model for *F* is used with RB_{phi} on the magnetic axis set to 0.17 Tm.
- Cubic model for pressure with u_0P on axis set to 0.1.
  - β=2/3
- $\tau_A = 1.88\text{x}10^{-7}$s
- 20x28 mesh with 4th order polynomial elements
- Equilibrium was designed with no open flux surfaces to aid simulations of nonlinear evolution.

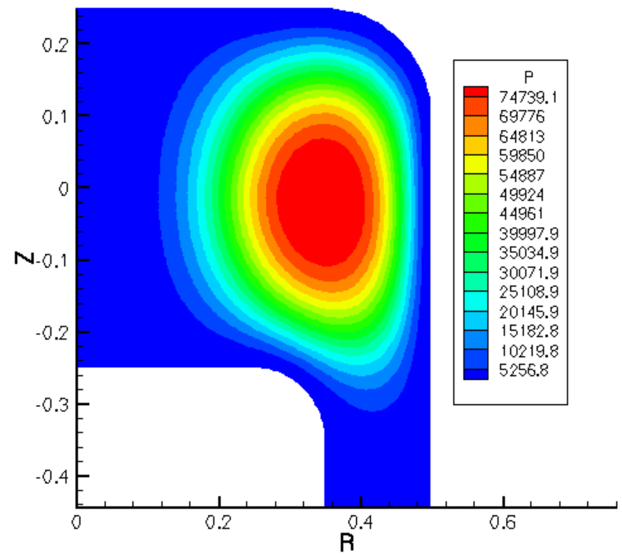


Axisymmetric nonlinear simulations of SSPX equilibrium test the quality of equilibrium.

Initial pressure



Final pressure



- Simulation is run for ~100 t_A
- Peak pressure changed by ~ 0.16%
- An isotropic number density diffusivity of 100 m^2/s is used for numerical stability.

Conclusions & Future Work

- A Grad-Shafranov solver has been developed for the NIMROD code.
- Fields created for a constant lambda cylindrical pinch show good agreement with analytical solution.
- Nonlinear simulations of equilibria ran over numerous Alfvén times suggest that the plasma is in equilibrium.
- Addition benchmarking is being performed.
- Functionality to create grid and generate initial magnetic field along the boundary will be created within NIMEQ.
- Use equilibria from NIMEQ to study SSPX and other configurations.